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2001 J. Phys. A: Math. Gen. 34 2179

(http://iopscience.iop.org/0305-4470/34/11/310)

J. Phys. A: Math. Gen. **34** (2001) 2179–2184 www.iop.org/Journals/ja PII: S0305-4470(01)15475-2

On integrability of the motion of an axisymmetric rigid body

Tatiana V Salnikova

Department of Mathematics and Mechanics, Moscow Lomonosov State University, Vorob'evy Gory, 119899 Moscow, Russia

Received 5 June 2000, in final form 22 November 2000

Abstract

We consider the problem of the existence of additional analytical first integrals in some Hamiltonian systems, which are close to integrable, namely, the motion of a rigid body is close to a dynamically symmetric one.

The problem of motion of a rigid body in an ideal liquid (the Kirchhoff problem) and the similar problem of rotation of a rigid body with a fixed point in an axisymmetric force field with a quadratic potential are investigated. The existence of hyperbolic periodic and asymptotic trajectories is shown. It is proved that perturbed trajectories are crossed but do not coincide. This is the reason for the absence of an additional analytical first integral in the perturbed problem.

The problem of perturbed motion of a dynamically symmetric rigid body along an absolutely smooth horizontal plane is considered. Non-integrability of this problem is proved by the method of splitting asymptotic surfaces.

PACS numbers: 4520J, 0230, 9510C

1. The Kirchhoff problem

The motion of a rigid body in an ideal fluid is described by the Kirchhoff equations in $R^6 = R^3\{M\} \times R^3\{e\}$:

$$\dot{M} = M \times \frac{\partial H}{\partial M} + e \times \frac{\partial H}{\partial e} \qquad \dot{e} = e \times \frac{\partial H}{\partial M}$$

where $H = \frac{1}{2}\langle AM, M\rangle + \langle BM, e\rangle + \frac{1}{2}\langle Ce, e\rangle$ is a positive-definite quadratic form, M is the kinetic moment and e is the impulsive force (see [1]). Matrix A can always be brought into diagonal form by means of an orthogonal transformation: $A = \mathrm{diag}(a_1, a_2, a_3)$, matrices B and C are symmetrical. If the body has three mutually perpendicular planes of symmetry, then $B = 0$, $C = \mathrm{diag}(c_{11}, c_{22}, c_{33})$.

The Kirchhoff equations always have three first integrals: $F_2 = \langle M, e\rangle$, $F_3 = \langle e, c\rangle$. If there exists an additional integral that is independent of the three classical ones, then the Kirchhoff equations are completely integrable.

As noted by Steklov in [2], in the case $B = 0$ the Kirchhoff equations are equivalent to the Euler–Poisson equations of motion of a rigid body with a fixed point in an axisymmetric force field with quadratic potential $-U(\gamma)$, where γ is the unit vector of the axis of symmetry of the force field. The Hamiltonian is

$$H = \tfrac{1}{2}\langle I^{-1}M, M\rangle - U(\gamma) = \tfrac{1}{2}\langle I\omega, \omega\rangle - U(\gamma).$$

The angular velocity vector is $\omega = \partial H/\partial M = AM = I^{-1}M$, where $1/a_i = I_i$, $i = 1, 2, 3$ are the principal central moments of inertia of some rigid body. In what follows, we will consider the case $B = 0$, and therefore we will not distinguish between the problem of rotation of a rigid body with a fixed point in an axisymmetric force field and the problem of motion of a rigid body in an ideal fluid. The equations of motion of a rigid body in a quadratic force field with potential

$$-U(\gamma_1, \gamma_2, \gamma_3) = \tfrac{1}{2}(c_{11}\gamma_1^2 + c_{22}\gamma_2^2 + c_{33}\gamma_3^2) + c_{12}\gamma_1\gamma_2 + c_{13}\gamma_1\gamma_3 + c_{23}\gamma_2\gamma_3$$

can be written in the form of the Euler–Poisson equations:

$$I\dot{\omega} + \omega \times I\omega = \gamma \times U'_\gamma \qquad \dot{\gamma} + \omega \times \gamma = 0.$$

If $c_{12} = c_{13} = c_{23} = 0$ and

$$(c_{22} - c_{33})/a_1 + (c_{33} - c_{11})/a_2 + (c_{11} - c_{22})/a_3 = 0$$

then these equations have a first integral (the case of Clebsch integrability). It was shown in [3] that if $a_1 \neq a_2 \neq a_3$, then no new integrable cases exist, except for the Clebsch case. If $a_1 = a_2 \neq a_3$ and $c_{11} = c_{22}$, then there exists a new integral $M_3 = \omega_3/a_3 =$ constant. This is the case of Kirchhoff integrability. From the standpoint of the problem of rotation of a rigid body with a fixed point, it is natural to also refer to this case as the Lagrange case.

Theorem. (See [4][1]). *Assume that $a_1 = a_2 \neq a_3$ and that the elements of the symmetrical matrix $C = \|c_{ij}\|$ are as follows: $c_{11} = c_{22} + \varepsilon$, $c_{22} \neq c_{33}$, εc_{12}, εc_{13}, εc_{23}. Then for small $\varepsilon \neq 0$ the Kirchhoff equations do not have a new integral that is analytic in R^6.*

Remark. If $a_1 = a_2 = a_3$, then the Kirchhoff equations are completely integrable.

The proof of the theorem is based on the Poincaré method of splitting the asymptotic surfaces [5, 6].

(a) Using the integral $F_3 = \langle\gamma, \gamma\rangle$, to within an additive constant we can represent the force functions in the following form:

$$U = \tfrac{1}{2}(c_{33} - c_{22})\gamma_3^2 + \tfrac{1}{2}\varepsilon\gamma_1^2 + \varepsilon(c_{12}\gamma_1\gamma_2 + c_{13}\gamma_1\gamma_3 + c_{23}\gamma_2\gamma_3).$$

Assume that $H = H_0 + \varepsilon H_1$ is the Hamiltonian of the problem of perturbed motion of a Lagrange top in a quadratic force field:

$$H_0 = \tfrac{1}{2}I_1(p^2 + q^2) + \tfrac{1}{2}I_3^2 r^2 + \tfrac{1}{2}(c_{33} - c_{22})\gamma_3^2$$
$$H_1 = \tfrac{1}{2}\gamma_1^2 + c_{12}\gamma_1\gamma_2 + c_{13}\gamma_1\gamma_3 + c_{23}\gamma_2\gamma_3.$$

Using the kinematic Euler equation and the area integral with constant equal to zero, we arrive at the unperturbed problem with one degree of freedom:

$$H_0 = \tfrac{1}{2}I_1\theta^2 + \widetilde{U} \qquad \widetilde{U} = \tfrac{1}{2}\cos^2\theta\left(\frac{I_3^2 r_0^2}{I_1 \sin^2\theta} + c_{33} - c_{22}\right).$$

[1] A similar theorem was formulated in [4] for the particular case in which $\|c_{ij}\|$ is a diagonal matrix.

(b) In order to obtain asymptotic solutions, let us determine the critical points at which $\widetilde{U}$ has a maximum, i.e. unstable equilibrium points. The point $\theta = \pi/2$ is always critical; the local maximum condition for $\widetilde{U}$ at point $\theta = \pi/2$ is

$$c_{33} - c_{22} < -\frac{I_3^2 r_0^2}{I_1} = -\alpha^2 \tag{1}$$

where α is a constant that depends on the initial conditions. For $c_{33} < c_{22}$ by choosing r_0 we can always achieve satisfaction of condition (1), i.e. an asymptotic trajectory runs through the point $\theta = \pi/2$ on the phase plane in this case. If $c_{33} > c_{22}$, then we replace γ by $\mathrm{i}\gamma$. This change does not affect the real property of the Kirchhoff equations, and corresponds to a simple change of sign of all the coefficients c_{ij}. After this change has been made, the condition $c_{33} < c_{22}$ will be satisfied. If there exists an integral of the Kirchhoff equations $F_4(\omega, \gamma)$, then the functions F_4' and F_4'', determined from the equation $F_4(\omega, \mathrm{i}\gamma) = F_4' + \mathrm{i}F_4''$, are integrals of a modified Kirchhoff system; at least one of them is independent with three classical integrals F_1, F_2, F_3. Thus, we can assume that the condition $c_{33} < c_{22}$ is always met and that asymptotic solutions run through the point $\theta = \pi/2$.

(c) In the homoclinic case, the splitting condition for the separatrices implies that the perturbed problem is not integrable [6, p 44]. The splitting condition is

$$\int_{-\infty}^{\infty} \{F_0, H_1\}(z_0(t), t) \not\equiv 0 \tag{2}$$

where $z_0(t)$ is the unperturbed asymptotic solution [6, p 42]). As the first integral of the unperturbed system with Hamiltonian H_0 we take the function $F_0 = \frac{1}{2}I_1(p^2+q^2)+\frac{1}{2}(c_{33}-c_{22})\gamma_3^2$, chosen from the convergence condition for integral (2). To calculate $\{F_0, H_1\}$ we employ the expressions

$$p = \frac{p_2\gamma_3 - p_3\gamma_2}{I_1} \qquad q = \frac{p_3\gamma_1 - p_1\gamma_3}{I_2} \qquad r = \frac{p_1\gamma_2 - p_2\gamma_1}{I_3}$$

where p_1, p_2, p_3 are the conjugate canonical momenta to γ_1, γ_2, γ_3 [7]. Then, for example,

$$\left\{F_0, \frac{\gamma_1^2}{2}\right\} = q\gamma_1\gamma_3 = -r_0\cos^2\theta\sin\varphi\cos\varphi\left(1 + \frac{I_3}{I_1\sin^2\theta}\right).$$

On the asymptotic trajectory

$$\varphi = r_0 t + \arctan\frac{\mathrm{e}^{2(a\beta t+k)} + 1 - 2a^2}{2a\sqrt{1-a^2}}$$

where $a^2 = 1 - I_3^2 r_0^2/I_1(c_2 - c_3)$, $\beta^2 = (c_2 - c_3)/I_1$, k is the initial phase. Note that the generality of the proof is not reduced by choosing the area constant in section 1 to be equal to zero, since it can be shown that the separatrices split under this condition, then the fact of continuity implies that they also split for sufficiently small values of the area constant. If no additional analytic integral exists in some region corresponding to a small interval of change of the area constant, then it does not exist in the entire region of variation of the parameters.

(d) Calculating (2) by means of residues, we obtain the non-splitting condition for the separatrices for $I_3 = 0$:

$$\frac{\beta}{\mathrm{e}^{\pi r_0/2} - \mathrm{e}^{-\pi r_0/2}}\sin 2r_0k - \frac{2\beta c_{12}}{\mathrm{e}^{\pi r_0/2} - \mathrm{e}^{-\pi r_0/2}}\cos 2r_0k + 4c_{13}\sin r_0k - 4c_{23}\cos r_0k \equiv 0.$$

A series of equalities follow from this: $c_{12} = c_{13} = c_{23} = 0$, $\beta = 0$. By the condition of the theorem, $c_{22} \neq c_{33}$, i.e. $\beta \neq 0$. Consequently, expression (2) is not identically zero and this proved that the perturbed problem is non-integrable for almost all $I_3 \neq 0$:

$$
\begin{aligned}
-\tfrac{1}{2} r_0 \int_{-\infty}^{\infty} \left(\cos^2\theta \sin 2\varphi + \frac{I_3}{I_1} \frac{\cos^2\theta}{\sin^2\theta} \sin 2\varphi \right) \mathrm{d}t \\
- r_0 c_{12} \int_{-\infty}^{\infty} \left(\cos^2\theta \cos 2\varphi + \frac{I_3}{I_1} \frac{\cos^2\theta}{\sin^2\theta} \cos 2\varphi \right) \mathrm{d}t \\
+ c_{13} \int_{-\infty}^{\infty} \left[\frac{\cos\theta}{\sin\theta} (1 - 2\cos^2\theta) \sqrt{\beta^2 - \frac{I_3^2 r_0^2}{I_1^2}} \, \frac{1 - \mathrm{e}^{1(a\beta t + k)}}{1 + \mathrm{e}^{2(a\beta t + k)}} \sin\varphi \right. \\
\left. - \frac{I_3}{I_1} r_0 \frac{\cos\theta}{\sin\theta} \cos^2\theta \cos\varphi \right] \mathrm{d}t + c_{23} \int_{-\infty}^{\infty} \left[\frac{\cos\theta}{\sin\theta} (1 - 2\cos^2\theta) \sqrt{\beta^2 - \frac{I_3^2 r_0^2}{I_1^2}} \right. \\
\left. \times \frac{1 - \mathrm{e}^{2(a\beta t + k)}}{1 + \mathrm{e}^{2(a\beta t + k)}} \cos\varphi + \frac{I_3}{I_1} r_0 \frac{\cos\theta}{\sin\theta} \cos^2\theta \sin\varphi \right] \mathrm{d}t \not\equiv 0.
\end{aligned}
$$

Upon the coefficients for the sines and cosines, for example, in the neighbourhood of $r_0 = 0$, we can readily show that they are not identically zero, and thus the theorem is proved in the general case.

2. A symmetric rigid body on a horizontal plane

Let a moving body of revolution be in contact with a fixed surface and P be the point of contact. In contrast to [8], we consider the case when the ellipsoid of inertia for the centre of mass G is an ellipsoid of revolution around the axis $G\bar{z}$. The distance from G to the plane is equal to $|GQ| = \zeta = f(\theta)$, where θ is the angle between the $\bar{z}$-axis and the vector GQ. Let $A = B \neq C$ be the principal central moments of inertia of the body.

Let us formulate the conditions of existence of asymptotic solutions for the unperturbed system with the Hamiltonian (see [9])

$$H_0 = \tfrac{1}{2} M \dot{\zeta}^2 + \tfrac{1}{2} A(p^2 + q^2) + \tfrac{1}{2} C r^2 + M g \zeta.$$

Using the kinematic Euler equations

$$p = \dot{\psi} \sin\theta \sin\varphi + \dot{\theta} \cos\varphi \qquad q = \dot{\psi} \sin\theta \cos\varphi - \dot{\theta} \sin\varphi$$

and the integrals of motion

$$
\begin{aligned}
&\tfrac{1}{2} M \dot{\zeta}^2 + \tfrac{1}{2} A(p^2 + q^2) + \tfrac{1}{2} C r^2 + M g \zeta = h = \text{constant} \\
&A p \sin\theta \sin\varphi + A q \sin\theta \cos\varphi + C r \cos\theta = K = \text{constant} \\
&r = r_0 = \text{constant}
\end{aligned}
$$

we obtain the equalities

$$[1 + c f'^2(\theta)] \dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta = \alpha - a f(\theta) \tag{3}$$

$$\dot{\psi} \sin^2\theta = \beta - b r_0 \cos\theta \qquad \dot{\varphi} = r_0 - \dot{\psi} \cos\theta \tag{4}$$

where $\alpha = \frac{1}{2}(2h - C r_0^2)$, $a = 2Mg/A$, $c = M/A$, $\beta = K/A$ and $b = C/A$. The equation

$$\dot{\theta}^2 [1 + c f'^2(\theta)] \sin^2\theta = [\alpha - a f(\theta)] \sin^2\theta - (\beta - b r_0 \cos\theta)^2 \tag{5}$$

follows from (3) and (4). For convenience, the following change is used: $\theta = \pi/2 + x$ and $f(\theta) = g(x)$. The initial conditions $\beta = 0$ and $\alpha = ag(0)$ correspond to the unperturbed periodic motion $\theta = \pi/2$, $\dot\theta = 0$, $\dot\psi = 0$, $\dot\varphi = r_0$. Using these initial conditions, from (5) we obtain

$$\dot x = F(x) = \frac{a[g(0) - g(x)]}{1 + cg'^2(x)} - \frac{b^2 r_0^2 \sin^2 x}{[1 + cg'^2(x)]\cos^2 x} \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

If $F''(0) > 0$, then the asymptotic trajectory passes through the point $x = 0$ on the phase plane. If $g(x)$ is an even function, then in a neighbourhood of the point $x = 0$ we have the expansion

$$g(x) = g(0) + \xi x^2 + \mathrm{o}(x^2)$$

where $\xi = g''(0)/2$. Hence, the condition $F''(0) > 0$ may be replaced by

$$\xi < -\frac{b^2 r_0^2}{a}.$$

Let us consider the perturbed problem (when the centre of mass is slightly displaced from the axis of symmetry along the $\overline{x}$-axis) with the Hamiltonian $H = H_0 + \varepsilon H_1$, where $H_1 = Mgl \sin\theta \sin\varphi$.

Let $f(\theta) = l \sin\theta$ ($g(x) = l\cos x$). This choice of f corresponds to the case when a disc of radius l slides along a horizontal plane. Then,

$$\dot x^2 = \frac{al(1 - \cos x)}{1 + cl^2 \sin^2 x} - \frac{b^2 r_0^2 \sin^2 x}{(1 + cl^2 \sin^2 x)^2 \cos^2 x}.$$

In order to simplify our calculations, we assume that $c \to 0$ and $b \to 0$ (i.e. $C \to 0$, $A \to 1$ and $M/A \to 0$). It follows from the analyticity that the result given below is valid for almost all values of the moment of inertia.

Now the equation for the asymptotic surface takes the form

$$\dot x^2 = al(1 - \cos x).$$

Following the method of splitting of asymptotic surfaces (see [6]), we calculate the Poisson bracket of the functions F_0 and H_1:

$$\{F_0, H_1\} = Mgl\dot x \sin x \sin\varphi.$$

Here $F_0 = H_0 - Cr_0^2/2$ is the first integral of the unperturbed system. Using the residues, we obtain

$$\int_{-\infty}^{\infty} \{F_0, H_1\}\big|_{ac}\, \mathrm{d}t = \frac{16\pi}{\exp\left(\pi r_0/\sqrt{2al}\right) - \exp\left(-\pi r_0/\sqrt{2al}\right)} \times \left[\frac{r_0}{\sqrt{2al}} \sin r_0 k - \left(3 + \frac{r_0^2}{2al}\right)\cos r_0 k\right]$$

where k is an arbitrary constant. This integral is not equal to zero identically. Hence, we proved that the asymptotic surfaces are split and that non-integrability of the perturbed problem takes place in the homoclinic case being considered (see [9]).

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